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An Experimental Determination of the Variation with Temperature of the Critical Velocity of Flow of Water in Pipes

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III. *An Experimental Determination of the Variation with Temperature of the Critical Velocity of Flow of Water in Pipes.*

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1. *Introduction.*

The motion of water in pipes and channels has been the subject of frequent investigation, both from the theoretical and the experimental side, and it is well known that while in some cases theory and experiment are in exact accord, yet in many others the experimental results differ widely from the calculated.

In some cases, while the theory holds for one set of conditions, it is found not to hold for conditions which at first do not appear to be fundamentally different.

A striking instance is that of the flow of a viscous liquid through a pipe of circular section, a case for which a strict mathematical solution can be obtained under certain assumed conditions of flow. Experiment shows that the theory is verified if the pipe is of capillary bore and the motion small, while if the pipe is large and the motion appreciable, there is a large discrepancy between experiment and calculation. The discrepancy is due to the assumption that the motion is stream-line, a condition of things which is true for tubes of capillary bore, but in general is not true for tubes of appreciable diameter unless the motion is below a certain limit, fixed by the size of the pipe and the physical characteristics of the liquid. Above this limit, the motion is eddying and the hydrodynamical equations no longer apply.

The change from stream-line to eddy or sinuous motion was first studied by Osborne Reynolds,* who showed that the determining factors in the case of a circular pipe depended on the dimensions of the pipe and the viscosity of the water. His results are based partly on deductions from the equations of motion for a viscous

* "An Experimental Investigation of the Circumstances which determine whether the Motion of Water shall be Direct or Sinuous, and of the Law of Resistance in Parallel Channels." 'Phil. Trans.,' 1883.

fluid; thus, if we take the general equations of motion for an incompressible fluid subject to no external forces, as of type

$$\frac{du}{dt} = -\frac{1}{\rho}\left\{\frac{d}{dx}(p_{xx} + \rho u^2) + \frac{d}{dy}(p_{yx} + \rho uv) + \frac{d}{dz}(p_{zx} + \rho uw)\right\}^*$$

and eliminate the pressures from the equations, we obtain the accelerations in terms of different types. Thus, if we take the middle term, viz., $-\frac{1}{\rho}\frac{d}{dy}(p_{yx} + \rho uv)$, and for p_{yx} write $\mu\left(\frac{dv}{dx} + \frac{du}{dy}\right)$, we get $-\frac{d}{dy}\left\{\frac{\mu}{\rho}\left(\frac{dv}{dx} + \frac{du}{dy}\right) + uv\right\}$. Now, since dv/dx and du/dy have the dimension of a velocity divided by a length and the other term has dimension of the square of a velocity, the relative values of these two terms are to one another as $\mu/c\rho$ to v, where c is a length, say the radius of the tube.

The equations do not show in what way the motion depends upon this relation, but it was inferred that the eddying motion must depend on some definite relation between v and $\mu/c\rho$, expressible in the form $v = k\mu/c\rho$, where k is some constant.

The experimental observations were of two kinds, the earlier depending on the device of introducing a colour band into a glass pipe and observing the velocity at which break-down of the stream-line motion occurred, and the later method depending upon the fact that stream-line motion is associated with resistance proportional to the velocity, while for eddy motion the resistance is proportional to a higher power of the velocity.

Both methods showed that the critical velocity at which stream-line motion changed to eddy motion varied directly as the viscosity, and inversely as the radius of the pipe.

Object of the Experiments.

In the experimental verification of the temperature effect upon the critical velocity a satisfactory agreement was obtained with the formula, but as the range of temperature was extremely limited, it was pointed out that "it would be desirable to make experiments at higher temperature; but there were great difficulties about this, which caused me, at all events for the time, to defer the attempt."†

It does not appear that such experiments have since been made, and although the difficulties were not estimated lightly, it seemed worth while to attempt experiments through a much larger range ot temperature.

Scope of the Experiments.

Although it would be eminently satisfactory to make experiments throughout the whole range of temperature of water, yet the experimental difficulties of maintaining

* 'Phil. Trans.,' A, 1895, p. 131.

† 'Phil. Trans.,' 1883, p. 977

a uniform temperature in the pipe increase in a much greater ratio than the increase of temperature beyond, say, 50° C., and there are other difficulties, due to convection and evaporation, which made it desirable to limit the investigation, at any rate for the time, to a range of 45° to 50° C. With these limits it was found that the decrease or increase of temperature along the pipe, when thickly lagged, was inconsiderable, and the correction to be applied was therefore small and not likely to cause an appreciable error. In order to carry on experiments at a higher temperature, it would apparently be necessary to surround the experimental tube with a water-jacket maintained at the same temperature as the water in the tank, otherwise drop of temperature along the pipe would be so considerable as to seriously increase the chances of error.

Method of Experiment.

The principle of the method is the same as originally devised by OSBORNE REYNOLDS, but the manner of carrying out the work differed somewhat in detail.

The method of colour bands is unsuitable for water at a high temperature, as it is impossible to eliminate the effect of conduction and convection, and the water consequently never comes to rest; moreover, experiments by this method lead to a different form of the criterion, viz., the maximum limit at which stream-line motion is possible, while experiments on the variation in the resistance of pipes lead to the minimum criterion, viz., that at which eddies change to steady motion. This latter method is also more likely to be accurate, for the maximum velocity of stream-line motion depends upon external causes, which influence it to a remarkable extent. Experiments were made with the tank in the laboratory to discover how far stream-line motion could be carried under favourable conditions; the tank rests directly upon the ground, and after water at the temperature of the room had been allowed to stand therein for two or three days, stream-line motion in pipes could be maintained at higher velocities than that given by the upper limit formula for the critical velocity v_c, viz.:

$$v_c = \frac{1}{43{\cdot}79}\frac{f(\tau)}{D},^{*}$$

the units being metres and degrees centigrade, a result no doubt due to the complete absence of vibration in the tank, which was founded on rock, and also the freedom of the water from sediment.

Moreover, it is easy to lower the critical velocity by subjecting the water to a disturbing cause; thus fine matter in suspension in the water will lower the critical velocity. Tapping the pipe or interposing therein a piece of wire gauze will also act likewise; in fact, the point of break-down can be varied within wide limits according to the circumstances.

Whatever be the disturbing causes, however, if stream-line motion exists, the

* 'Phil. Trans.,' 1884, p. 957.

relation of slope to velocity is a perfectly definite one at a definite temperature for the flux, being expressed by the equation

$$q = \frac{\pi r^4}{8\mu}\left(\frac{p_1 - p_2}{l}\right).*$$

If we write $\bar{v} = \frac{q}{\pi r^2}$ and $\frac{p_1 - p_2}{l} = i$, we obtain $\bar{v} = \frac{r^2}{8\mu}i$, and this relation between $\bar{v}$ and i, plotted logarithmically, is, at a definite temperature, a line inclined at 45° to the axes.

Slightly above critical velocity, it can be shown experimentally that no definite relation exists, but well above this point, where the motion is perfectly eddying, it can be shown experimentally that the relation between $\bar{v}$ and i is a perfectly definite one at a definite temperature, and is expressed by some straight line inclined at an angle $\tan^{-1} n$, where n is a constant for any particular pipe.

It therefore appears that the minimum critical velocity is the intersection of the two branches of the logarithmic homologue; and throughout this paper this point has been taken as the critical velocity for the temperature considered.

As the experiments below the critical velocity require apparatus for measuring pressures of extreme accuracy and limited range, while above the critical velocity the limit of accuracy is relatively less important and the range is large, it simplifies matters to take a series of runs at different temperatures below the critical velocity without any change, and afterwards to take runs above the critical velocity. With this method the variation of temperature during a single series is small, and the correction to a standard temperature is generally negligible.

Apparatus used in the Experiments.

The experimental tank A, fig. 1, is of cast-iron, 5 feet square in section and about

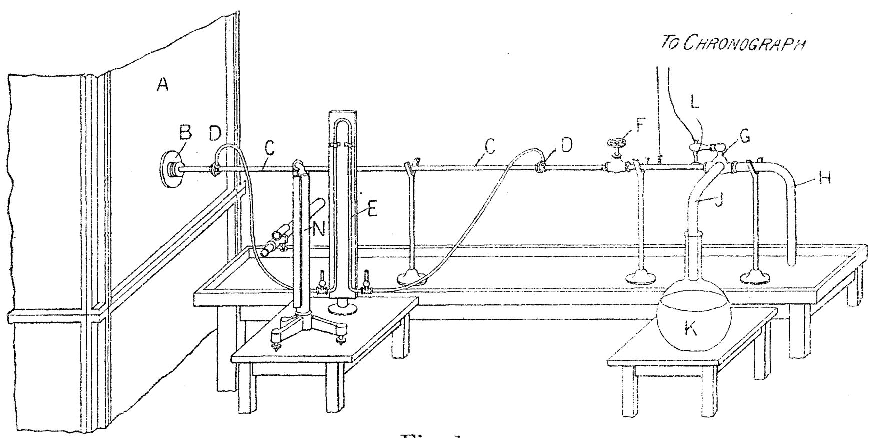

Fig. 1.

30 feet in height, its base resting upon the earth, so that the water in it is not easily disturbed by external causes. It is provided with a steam heater for the inflowing

* LAMB'S 'Hydrodynamics,' p. 521.

water, and there is a direct steam connection to the boiler room, so that steam can be blown directly into the tank. About 8 feet above the base there is an opening, B, in the middle of one side, through which the tube C was inserted, its bell-mouth being placed at the centre; and at suitable distances apart pressure chambers, D, were formed and connected up to the U-gauge E. The flow of water was controlled by a valve, F, and on the prolongation of the pipe a three-way plug valve, G, was inserted, so that the water could run to waste through the pipe H, or could be discharged, by the pipe J, into the glass flask K. The handle of this tap was provided with a flexible brass plate, L, in circuit with a chronograph, so that at the middle of its swing a circuit was completed by the contact of the brass strip with the pipe, and a record was obtained on the drum of the chronograph. This latter instrument was furnished with two pens, marking in opposite directions, one ticking seconds and the other operating at the beginning and end of each run. This arrangement tends to prevent errors in reading.

The pressure chambers were of a special design and consisted of three separate pieces, the outer one (A) of which couples the parts B and C together, leaving a continuous opening, D, which may be of any required width. In the present case, the two sides forming the slit were separated by an interval not more than $\frac{1}{200}$ inch, so as to prevent, as far as possible, any interference with stream-like flow. The part B is recessed to form a pressure chamber, P, connected to the gauge by an opening, E. The parts B and C are faced so that when drawn together by the coupling A, they form a water-tight joint at F, and the ends of the pipe are screwed into corresponding recesses in B and C. This form of pressure chamber has several advantages. The continuous opening gives an accurate mean value of the pressure, and it can be faced without any burr; moreover, it may be readily disconnected for inspection.

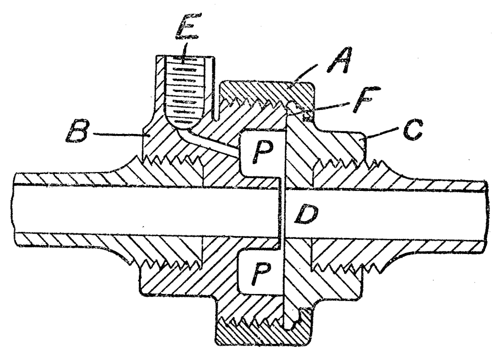

Fig. 2.

The pipe was of brass, without seam, and 6 feet in length between the pressure chambers; its mean diameter was determined by first weighing empty and then full of mercury. The mean diameter thus determined was 0·3779 inch.

The Measurement of Pressure.

The accurate determination of the pressures at the given sections of the pipe is a matter of considerable difficulty, especially at the very low differences of head required for the accurate determination of the slope of pressure at velocities below the critical velocity. At the higher pressures, a U-tube containing mercury was found to answer all requirements. Errors due to the inequalities of the tube were got rid of by measurements taken on both tubes, while a suitable correction was made for temperature. At the low velocities, considerable difficulties were experienced. The difference of heads between the two sections at the lowest velocities was about ·005 inch of mercury, and as this must be read very accurately, it became a matter of such difficulty that a new gauge was made and filled with carbon bisulphide. A number of trials were made, but it was found that the carbon bisulphide was very sluggish in action, and, unless a very considerable time was allowed between every two successive runs, its readings could not be relied upon. Another and more serious defect was the shape of the falling meniscus, which rarely assumed its proper form, so as to afford a definite measurement. This was due to the adherence of the carbon bisulphide to the glass; and in spite of repeated cleanings with different re-agents, no decided improvement was made and the gauge was abandoned. A return was made to the mercury gauge, and the cathetometer was replaced by micrometer microscopes, which had been carefully calibrated beforehand. These afforded much better results, but the observations were still irregular. Finally, the solution of the difficulty was found by turning the U-gauge upside down and imprisoning in its upper part a fixed column of air above the water in both limbs of the gauge.

At first sight this might not seem to be a good arrangement, since any small variation of temperature will affect the imprisoned volume of air considerably, but this affects both legs equally, and there is no error from this cause. A possible source of error is the creeping of air from the pipe to the gauges. This is extremely unlikely, as the air in this case must first descend. If any liberation of air occurred from the water, its effect in altering the gauge would only be momentary.

In practice, this gauge proved extremely sensitive and the readings could be repeated very accurately.

The cathetometer used in reading the heights of the liquid in the U-tubes was of a standard pattern made by the Cambridge Scientific Instrument Company and capable of reading to $\frac{1}{100}$ of a millimetre.

Stream-line Flow.

The determination of the relation of slope to pressure, for water in stream-line motion flowing through tubes of more than capillary size, is rendered somewhat difficult because of the smallness of the difference of pressure required to produce the flow. The difference may be increased by using a long length of pipe, or by using apparatus of extreme accuracy. The disposition of the permanent apparatus in the laboratory prevented the use of a pipe more than 6 feet in length between the pressure chambers; and at first considerable difficulty was experienced in obtaining consistent results, but after many trials this was accomplished. A disturbing cause, which could not be altogether avoided, was the rise or fall of the temperature of the water as it flowed along the pipe; a fall if the temperature of the room was above the temperature of the water, and *vice versâ*. This was partly removed by covering the pipe with thick cotton-wool lagging overlaid with flannel, and in order to obtain a mean value of the temperature of the water in the pipe a long-stem thermometer was fixed in the tank and another was immersed in the outflowing water, and a mean value of these readings was taken as the true temperature in the pipe.

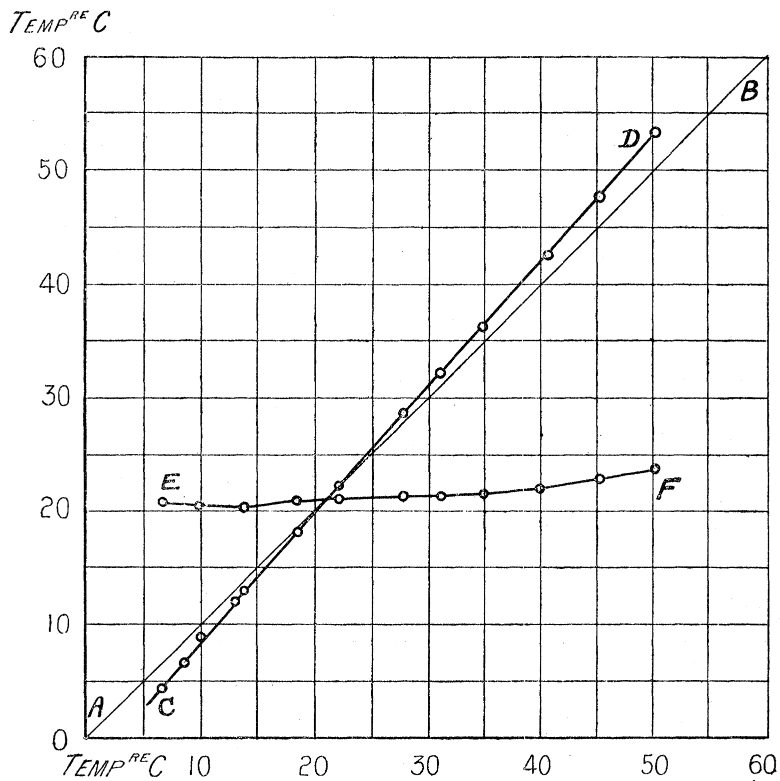

Fig. 3.

A plot of the variations obtained is shown in fig. 3, in which the line AB gives the temperature of the outflow water, CD the temperature of the tank, and EF the corresponding temperature of the room. The relation between the tank temperature and outflow temperature is shown to be practically a linear one, thereby warranting the correction.

In all, ten series of runs were made at temperatures covering the range, and the results obtained are recorded in the following table, and are shown on fig. 4.

TABLE I.

Number of experiment.	Temperature, ° C.	Time, seconds.	Total weight of water discharged, pounds.	Difference of head in centimetres of water.	log v, v in feet per second.	log h', h' in feet of water.
1	4·3	121·28	7·112	3·819	0·0815	$\bar{1}$·0975
2	4·3	188·84	9·402	3·209	0·0102	$\bar{1}$·0219
3	4·4	148·60	6·312	2·713	$\bar{1}$·9412	$\bar{2}$·9491
4	4·5	137·71	4·204	1·921	$\bar{1}$·7978	$\bar{2}$·7991
5	4·6	164·92	4·033	1·539	$\bar{1}$·7016	$\bar{2}$·7028
6	4·6	198·80	3·380	1·067	$\bar{1}$·5436	$\bar{2}$·5438
7	4·7	84·26	1·336	0·999	$\bar{1}$·5134	$\bar{2}$·5152
8	4·7	152·00	1·540	0·648	$\bar{1}$·3189	$\bar{2}$·3272
9	11·2	141·18	6·130	2·315	$\bar{1}$·9511	$\bar{2}$·8799
10	11·2	168·15	5·131	1·570	$\bar{1}$·7978	$\bar{2}$·7133
11	11·2	194·18	6·686	1·811	$\bar{1}$·8503	$\bar{2}$·7733
12	11·2	140·40	3·318	1·224	$\bar{1}$·6868	$\bar{2}$·6032
13	11·2	199·70	3·018	0·795	$\bar{1}$·4925	$\bar{2}$·4158
14	11·3	235·68	2·407	0·507	$\bar{1}$·3224	$\bar{2}$·2204
15	11·3	175·10	1·295	0·388	$\bar{1}$·1824	$\bar{2}$·1042
16	16·8	70·90	1·991	1·221	$\bar{1}$·7621	$\bar{2}$·6020
17	16·8	82·00	3·138	1·678	$\bar{1}$·8964	$\bar{2}$·7401
18	16·8	75·70	3·565	2·088	$\bar{1}$·9865	$\bar{2}$·8351
19	16·8	83·53	2·266	1·212	$\bar{1}$·7470	$\bar{2}$·5988
20	16·8	88·28	1·320	0·660	$\bar{1}$·4883	$\bar{2}$·3348
21	16·8	180·43	2·707	0·668	$\bar{1}$·4898	$\bar{2}$·3401
22	16·9	143·75	1·347	0·404	$\bar{1}$·2855	$\bar{2}$·1217
23	16·9	123·00	4·439	1·594	$\bar{1}$·8710	$\bar{2}$·7178
24	18·0	188·60	7·150	1·671	$\bar{1}$·8825	$\bar{2}$·7380
25	18·0	159·35	4·821	1·304	$\bar{1}$·7945	$\bar{2}$·6302
26	18·0	182·55	3·973	0·932	$\bar{1}$·6515	$\bar{2}$·4844
27	18·1	187·30	3·926	0·931	$\bar{1}$·6353	$\bar{2}$·4839
28	18·1	190·15	2·713	0·604	$\bar{1}$·4682	$\bar{2}$·2960
29	18·1	181·53	1·208	0·183	$\bar{1}$·1368	$\bar{3}$·7775
30	18·1	181·30	3·790	0·886	$\bar{1}$·6340	$\bar{2}$·4624
31	18·2	111·25	1·155	0·456	$\bar{1}$·3301	$\bar{2}$·1740
32	18·2	346·40	10·932	1·376	$\bar{1}$·8129	$\bar{2}$·6536
33	27·2	135·85	4·594	1·232	$\bar{1}$·8445	$\bar{2}$·6059
34	27·2	166·33	4·782	1·020	$\bar{1}$·7733	$\bar{2}$·5239
35	27·2	135·12	2·929	0·779	$\bar{1}$·6508	$\bar{2}$·4068
36	27·2	177·30	3·140	0·660	$\bar{1}$·5629	$\bar{2}$·3348
37	27·1	195·48	2·294	0·443	$\bar{1}$·3842	$\bar{2}$·1617
38	27·1	139·78	6·300	1·674	$\bar{1}$·9684	$\bar{2}$·7391
39	27·1	83·33	4·404	2·028	0·0378	$\bar{2}$·8224
40	31·1	175·83	8·218	1·628	$\bar{1}$·9848	$\bar{2}$·7263
41	31·1	123·95	4·346	1·172	$\bar{1}$·8601	$\bar{2}$·5837
42	31·1	146·55	3·717	0·851	$\bar{1}$·7194	$\bar{2}$·4446
43	31·1	150·30	1·334	0·304	$\bar{1}$·2635	$\bar{3}$·9976

TABLE I.—*continued.*

Number of experiment.	Temperature, ° C.	Time, seconds.	Total weight of water discharged, pounds.	Difference of head in centimetres of water.	log v, v in feet per second.	log h', h' in feet of water.
44	34·5	157·70	4·329	0·871	$\bar{1}$·7541	$\bar{2}$·4553
45	34·5	105·60	5·563	1·713	·0372	$\bar{2}$·7490
46	34·4	96·68	3·745	1·214	$\bar{1}$·9037	$\bar{2}$·5995
47	34·4	158·10	4·122	0·767	$\bar{1}$·7317	$\bar{2}$·4001
48	34·3	222·02	3·487	0·463	$\bar{1}$·5117	$\bar{2}$·1809
49	34·2	138·63	2·105	0·458	$\bar{1}$·4969	$\bar{2}$·1762
50	34·2	301·23	3·551	0·357	$\bar{1}$·3870	$\bar{2}$·0680
51	38·0	251·10	9·849	1·179	$\bar{1}$·9098	$\bar{2}$·5866
52	38·0	141·37	6·872	1·477	·0031	$\bar{2}$·6844
53	37·9	112·45	3·893	1·027	$\bar{1}$·8556	$\bar{2}$·5265
54	37·8	265·68	6·681	0·719	$\bar{1}$·7169	$\bar{2}$·3717
55	37·8	194·45	3·518	0·497	$\bar{1}$·5738	$\bar{2}$·2114
56	37·7	162·95	2·043	0·371	$\bar{1}$·4144	$\bar{2}$·0844
57	37·7	148·73	2·318	0·442	$\bar{1}$·5089	$\bar{2}$·1604
58	42·4	140·27	5·365	1·031	$\bar{1}$·8998	$\bar{2}$·5281
59	42·4	121·07	3·532	0·779	$\bar{1}$·7820	$\bar{2}$·4064
60	42·3	130·38	2·879	0·583	$\bar{1}$·6613	$\bar{2}$·2806
61	42·3	158·38	2·720	0·474	$\bar{1}$·5520	$\bar{2}$·1907
62	42·2	229·33	3·384	0·397	$\bar{1}$·4860	$\bar{2}$·1137
63	42·2	167·63	1·825	0·294	$\bar{1}$·3541	$\bar{3}$·9832
64	42·1	124·75	4·365	0·933	$\bar{1}$·8611	$\bar{2}$·4848
65	49·5	213·65	3·673	0·417	$\bar{1}$·5539	$\bar{2}$·1351
66	49·4	268·15	7·015	0·637	$\bar{1}$·7362	$\bar{2}$·3191
67	49·3	162·93	5·766	0·836	$\bar{1}$·8745	$\bar{2}$·4372
68	49·0	303·75	1·929	0·153	$\bar{1}$·1214	$\bar{3}$·6997
69	48·9	134·80	2·383	0·423	$\bar{1}$·5660	$\bar{2}$·1413
70	48·8	162·05	3·839	0·578	$\bar{1}$·6930	$\bar{2}$·2769
71	48·7	139·88	4·692	0·794	$\bar{1}$·8440	$\bar{2}$·4148

In this table the observations are recorded in Columns 2, 3, 4 and 5, and from Columns 3 and 4 the mean values of the velocity of the water in feet per second have been calculated, and the logarithms of these quantities are given in Column 6. The observed differences of head given in Column 5 have been reduced to feet of water, h', to correspond, and the values of log h' are given in Column 7.

In most cases, owing to the large volume of water in the tank (usually not less than 300 cubic feet), the temperature remained remarkably steady during the runs forming a series, and no correction for temperature was necessary, and none was made unless the temperature differed more than 0°·1 C. In some cases, however, a much greater variation was met with, especially at the higher temperatures, and correction was necessary, not only in this series, but in the second series when the

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motion was eddy or sinuous. The correction factor to be applied may be obtained as follows:—

If we assume that in stream-line motion or sinuous motion the total resistance i depends on powers of the pipe radius, the kinematic viscosity, the density and the velocity, we may write, with the usual notation,

$$i = kr^x \nu^y \rho^z v^n,$$

where r = radius of the pipe,
ν = coefficient of kinematic viscosity,
ρ = density,
$\bar{v}$ = mean velocity of water along the pipe,
k = a constant.

Dimensionally this equation becomes

$$\frac{[\mathrm{M}]\,[\mathrm{L}]}{[\mathrm{T}^2]} = [\mathrm{L}]^x \left[\frac{\mathrm{L}^2}{\mathrm{T}}\right]^y \left[\frac{\mathrm{M}}{\mathrm{L}^3}\right]^z \left[\frac{\mathrm{L}}{\mathrm{T}}\right]^n,$$

[giving the relations

$$z = 1, \qquad x + 2y - 3z + n = 1, \qquad y + n = 2,$$

and therefore

$$i = k\rho r^n \nu^{2-n} v^n.$$

For the case of stream-line motion, $n = 1$ giving

$$i = k\rho r \nu v.$$

For the case of sinuous motion n is greater than unity, and we may write the equation

$$i = k\rho \frac{\mu^{2-n}}{\rho^{2-n}} r^n v^n$$

$$= \mathrm{K}\mu^{2-n}\rho^{n-1}, \qquad \text{where } \mathrm{K} = kv^n r^n.$$

Taking logarithms, we get

$$\log i = \log \mathrm{K} + (2 - n) \log \mu + (n - 1) \log \rho.$$

Differentiating with v constant, we obtain

$$\frac{1}{i}\frac{di}{d\tau} = (2 - n)\frac{1}{\mu}\frac{d\mu}{d\tau} + (n - 1)\frac{1}{\rho}\frac{d\rho}{d\tau}.$$

Now $\mu = \dfrac{c}{1 + \alpha\tau + \beta\tau^2}$, therefore $\dfrac{1}{\mu}\dfrac{d\mu}{d\tau} = \dfrac{-(\alpha + 2\beta\tau)}{1 + \alpha\tau + \beta\tau^2}$, and $\rho = \rho_0 (1 - \gamma t)$ approximately, therefore

$$\frac{1}{\rho}\frac{d\rho}{d\tau} = -\frac{\gamma}{1-\gamma\tau}.$$

Hence

$$\frac{di}{i} = -(2-n)\frac{\alpha+2\beta\tau}{1+\alpha\tau+\beta\tau^2}d\tau + (1-n)\frac{\gamma}{1-\gamma\tau}d\tau.]^*$$

For stream-line motion $n = 1$, and the values of α and β taken from POISEUILLE'S formula are

$$\alpha = \cdot 03368, \qquad \beta = \cdot 000221.$$

The value of γ for water is approximately ·00048, and hence the second term is never important, and the first term is only important where $d\tau$ is large (and the term $\alpha + 2\beta\tau$ is not small).

For stream-line motion the correction for a difference of 1° at 10° C. is approximately ·028, while at 50° C. it is ·017.

The observations recorded in the table, corrected to a mean temperature, are plotted in fig. 4. [†The diagram shows the logarithmic homologues for stream-line motion in the pipe at ten different temperatures between 4° C. and 50° C., and these are represented by a series of straight lines equally inclined to the axes of co-ordinates. The mean temperatures to which the observations have been reduced are as follows:—

Experiments.	Reduced to a mean temperature Centigrade of	Corresponding lines on fig. 4.
1 to 8	4·5°	1
9 „ 15	11·2	2
16 „ 23	16·8	3
24 „ 32	18·1	4
33 „ 39	27·2	5
40 „ 43	31·1	6
44 „ 50	34·4	7
51 „ 57	37·8	8
58 „ 64	42·3	9
65 „ 71	49·3	10

These lines were] found to agree closely with the formula

$$q = \frac{\pi r^4}{8\mu}\cdot\frac{p_1 - p_2}{l},\ddagger$$

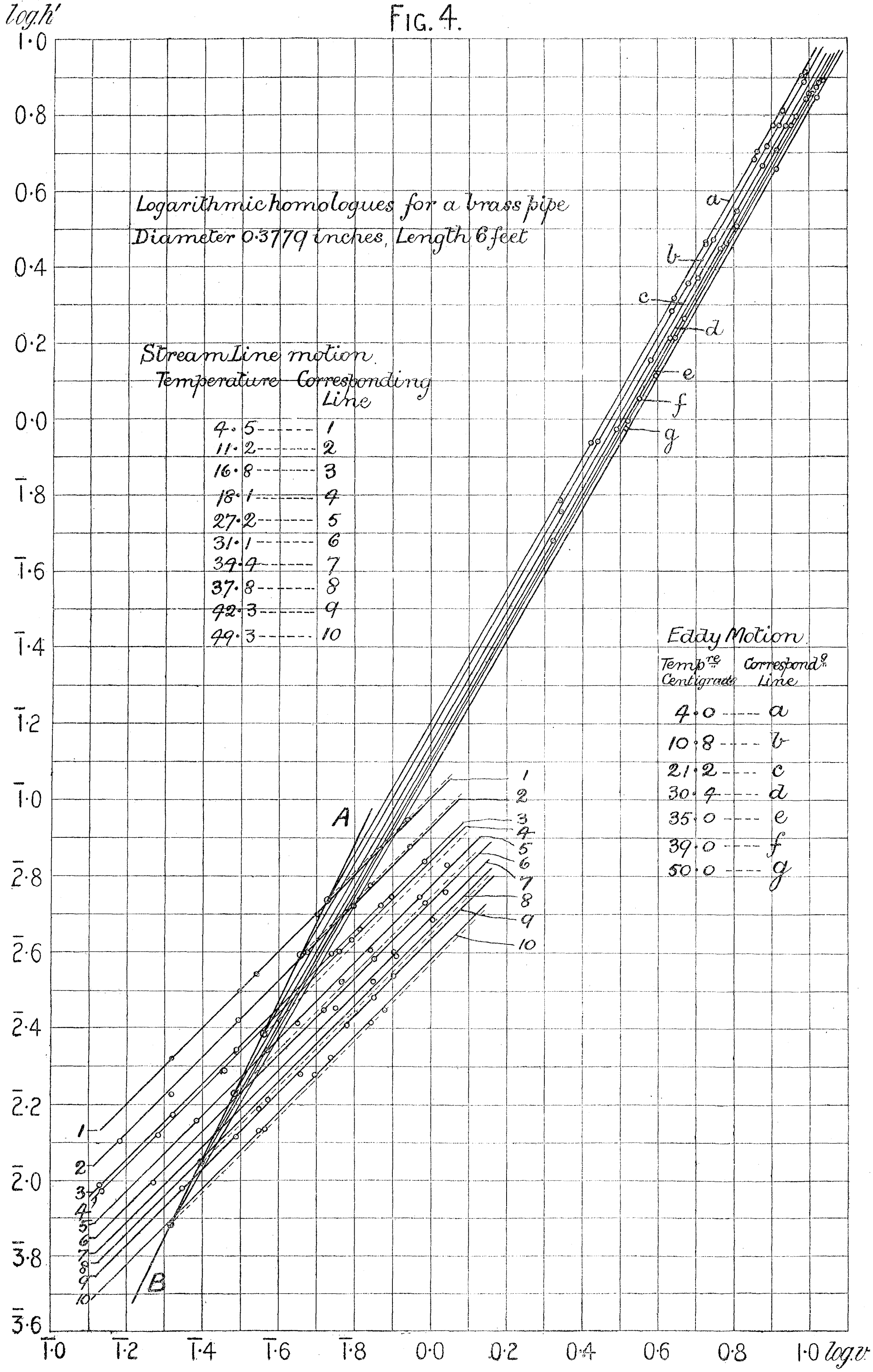
where r is the radius of the pipe, p_1 and p_2 the pressures at the ends, l the length of pipe, and μ is the coefficient of viscosity.

* Corrected Nov. 14, 1902, as pointed out by the Referee.

† Added Nov. 14, 1902.

‡ *Loc. cit. ante* (p. 48).

FIG. 4.



log h′
Logarithmic homologues for a brass pipe
Diameter 0·3779 inches, Length 6 feet
Stream Line motion
Temperature Corresponding Line
4·5 ------ 1
11·2 ------ 2
16·8 ------ 3
18·1 ------ 4
27·2 ------ 5
31·1 ------ 6
34·4 ------ 7
37·8 ------ 8
42·3 ------ 9
49·3 ------ 10
Eddy Motion
Temp.re Centigrade Correspond.g Line
4·0 ---- a
10·8 ---- b
21·2 ---- c
30·4 ---- d
35·0 ---- e
39·0 ---- f
50·0 ---- g
A
B
log v

This agreement is shown clearly by fig. 5, where the intersections of the logarithmic homologues with the zero ordinate are plotted with reference to the temperature as abscissa, and are compared with the intersections determined from the equation above, taking the value of μ according to POISEUILLE'S formula. At temperatures between 5° and 20° C. the agreement is close, the values at 27°·2 and 31° do not correspond very well, and there is a very fair agreement at the higher temperatures. The dotted lines on fig. 4 are the logarithmic homologues at the temperatures of 4°, 10°·8, 21°·2, 30°·4, 35°, 39° and 50° C. respectively, and these have been interpolated by aid of figs. 4 and 5, in order to determine the intersections with the homologues for eddy motion also plotted on fig. 4, and which are referred to in the next section.

The Relation of Slope to Velocity for Water in Eddy Motion.

A second series of experiments was now commenced with water in eddy motion to determine the relation between the loss of head and the velocity at a sufficient number of temperatures within the range.

It was extremely difficult to control the temperature, and so no attempt was made to obtain a series of runs with temperatures corresponding exactly to those obtained

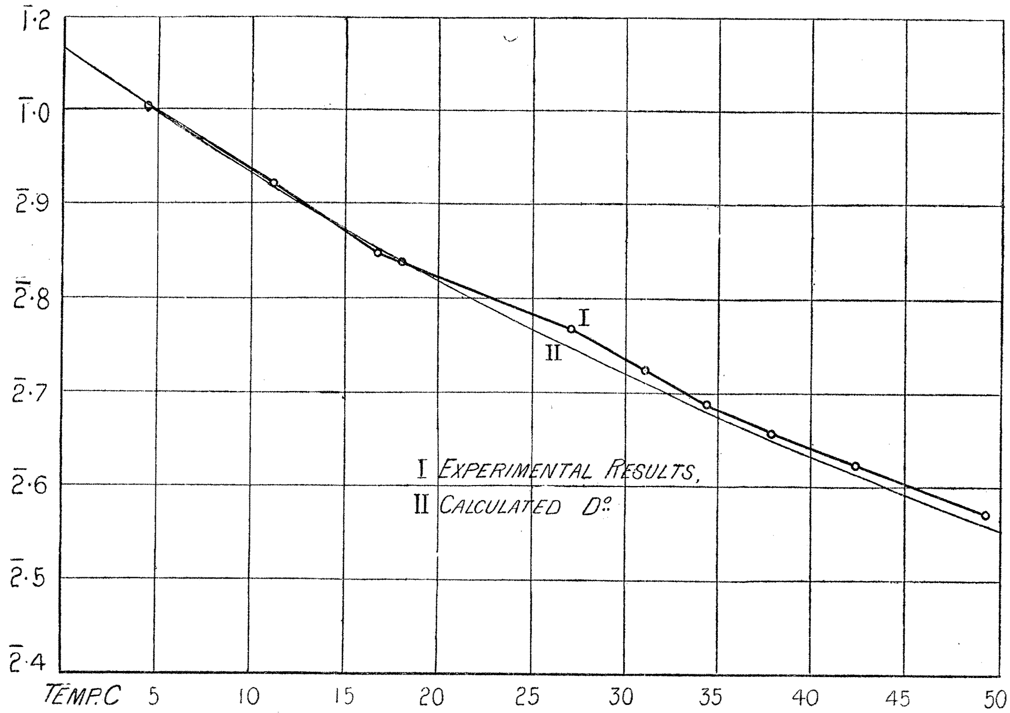


Fig. 5.

for stream-line motion, nor was this necessary, as the logarithmic homologue for stream-line motion, corresponding to a similar one for eddy motion, was obtained by interpolation from figs. 4 and 5. The observations were made under precisely the same conditions as before, except that in the pressure gauge mercury in contact with water was used instead of water in contact with air and the gauge was in its normal position with the connecting U below.

After some preliminary trials, seven complete series of runs were made, covering the range of temperature, and the observations made are recorded in Table II., and the logarithmic plots are shown on fig. 4.

As slight differences of temperature were now of much less importance, the plotted results lie much better upon the mean line, and enable the value of n to be determined with considerable accuracy.

TABLE II.

Number of experiment.	Temperature, ° C.	Time, seconds.	Total weight of water discharged, pounds.	Difference of head in centimetres of mercury.	log v, v in feet per second.	log h', h' in feet of water.
72	4·0	25·05	11·849	18·313	·9882	·9119
73	4·0	26·00	10·791	14·631	·9313	·8144
74	4·0	36·20	12·873	11·353	·8642	·7042
75	4·0	38·48	13·559	11·039	·8602	·6920
76	4·0	49·30	12·780	6·520	·7270	·4633
77	4·0	62·05	13·198	4·612	·6410	·3130
78	4·0	98·08	12·603	1·945	·4221	$\bar{1}$·9380
79	4·1	27·50	12·868	18·028	·9834	·9051
80	4·1	34·83	13·665	13·327	·9068	·7738
81	4·1	30·68	11·999	13·271	·9054	·7720
82	10·8	31·10	12·605	13·422	·9224	·7763
83	10·8	46·80	12·748	6·719	·7484	·4758
84	10·8	62·95	13·257	4·300	·6368	·2820
85	10·8	120·60	13·235	1·392	·3537	$\bar{1}$·7921
86	10·8	25·86	12·221	17·469	·9878	·8908
87	10·9	34·74	13·131	11·852	·8908	·7223
88	10·9	5·58	13·006	5·137	·6808	·3592
89	10·9	100·00	13·405	1·967	·4406	$\bar{1}$·9423
90	21·2	28·38	12·037	13·481	·9416	·7774
91	21·2	34·62	12·864	10·701	·8841	·6771
92	21·2	39·58	12·444	7·942	·8815	·5476
93	21·2	51·27	12·668	5·280	·7068	·3703
94	21·2	70·00	13·094	3·228	·5859	·1567
95	21·2	114·90	12·797	1·265	·3609	$\bar{1}$·7498
96	21·2	32·82	12·894	11·882	·9803	·7227
97	30·4	26·81	13·250	16·206	1·0090	·8573
98	30·4	24·66	11·960	16·145	1·0009	·8556
99	30·4	28·93	12·679	13·536	·9569	·7791
100	30·4	28·49	12·851	14·138	·9693	·7979
101	30·4	31·75	12·639	11·472	·9151	·7072
102	30·4	45·64	12·908	6·326	·7665	·4487
103	30·4	61·05	12·870	3·743	·6390	·2208
104	30·4	83·60	12·580	2·122	·4926	$\bar{1}$·9743
105	30·4	127·70	12·968	1·082	·3218	$\bar{1}$·6818
106	30·4	25·11	11·995	15·725	·9942	·8442

TABLE II.—*continued.*

Number of experiment.	Temperature, ° C.	Time, seconds.	Total weight of water discharged, pounds.	Difference of head in centimetres of mercury.	log v, v in feet per second.	log h', h' in feet of water.
107	35·0	23·05	11·898	17·415	1·0285	·8888
108	35·0	22·52	11·611	17·397	1·0277	·8884
109	35·0	27·90	11·854	12·703	·9440	·7518
110	35·0	42·00	13·034	7·213	·8075	·5060
111	35·0	54·95	12·508	4·231	·6725	·2743
112	34·9	73·40	12·527	2·541	·5475	·0529
113	34·9	26·10	13·370	17·220	1·0252	·8839
114	40·4	23·57	12·311	17·482	1·0343	·8905
115	39·6	25·60	10·905	11·792	·9458	·7195
116	39·2	40·50	11·869	6·526	·7834	·4624
117	38·8	56·55	12·214	3·770	·6508	·2241
118	38·5	77·14	12·308	2·228	·5191	$\bar{1}$·9957
119	38·0	24·67	12·802	17·550	1·0313	·8920
120	50·5	24·76	12·531	15·820	1·0226	·8470
121	50·5	67·20	12·771	2·933	·5972	·1151
122	50·0	27·18	12·245	12·708	·9721	·7519
123	50·0	56·48	11·870	3·340	·6410	·1716
124	50·0	32·40	12·928	10·492	·9194	·6686
125	49·8	78·40	12·346	2·147	·5178	$\bar{1}$·9796

The slopes of the lines for the different temperatures are shown in the accompanying table, and their mean value is $n = 1·731$.

Temperature, ° C.	4·0	10·8	21·2	30·4	35·0	39·0	50·0
n	1·722	1·733	1·740	1·734	1·738	1·737	1·715

The different values obtained are no doubt due to temperature errors, and this view is confirmed when it is seen that the variations from the mean value are greater the further the temperature of the water is from the temperature of the laboratory.

As there seems no reason to suppose that the value of n varies with the temperature, the logarithmic homologues have been drawn at a mean inclination of $\tan^{-1} 1·731$ in determining the critical velocity. The corrections to be applied for differences of temperature are now much smaller, and for the mean value of $n = 1·731$ is found to be ·0076 for a difference of 1° from a temperature of 10° C., and ·0047 for the same difference at 50° C. The observations for each series of runs in Table II. have been plotted to a mean temperature like those for stream-line motion, the temperatures corresponding to the interpolated homologues for stream-line motion described above.

Critical Velocity.

It has been pointed out in an earlier section that no attempt was made to determine the velocity at which stream-line motion broke down, but that the intersections of the two sets of lines above and below critical velocity were used to determine the minimum critical velocity. This method of procedure amounts to the determination of the curve of intersection of two families of straight lines, whose positions are experimentally determined, and it is clear that if the points of intersection lie upon some straight line in the logarithmic plot, the variation of the critical velocity must follow the viscosity of the water linearly, while, if they do not, the law cannot be a linear one.

Fig. 4 shows the observations for stream-line flow, and the lines representing eddy motion are drawn thereon, and are produced to meet the interpolated lines for

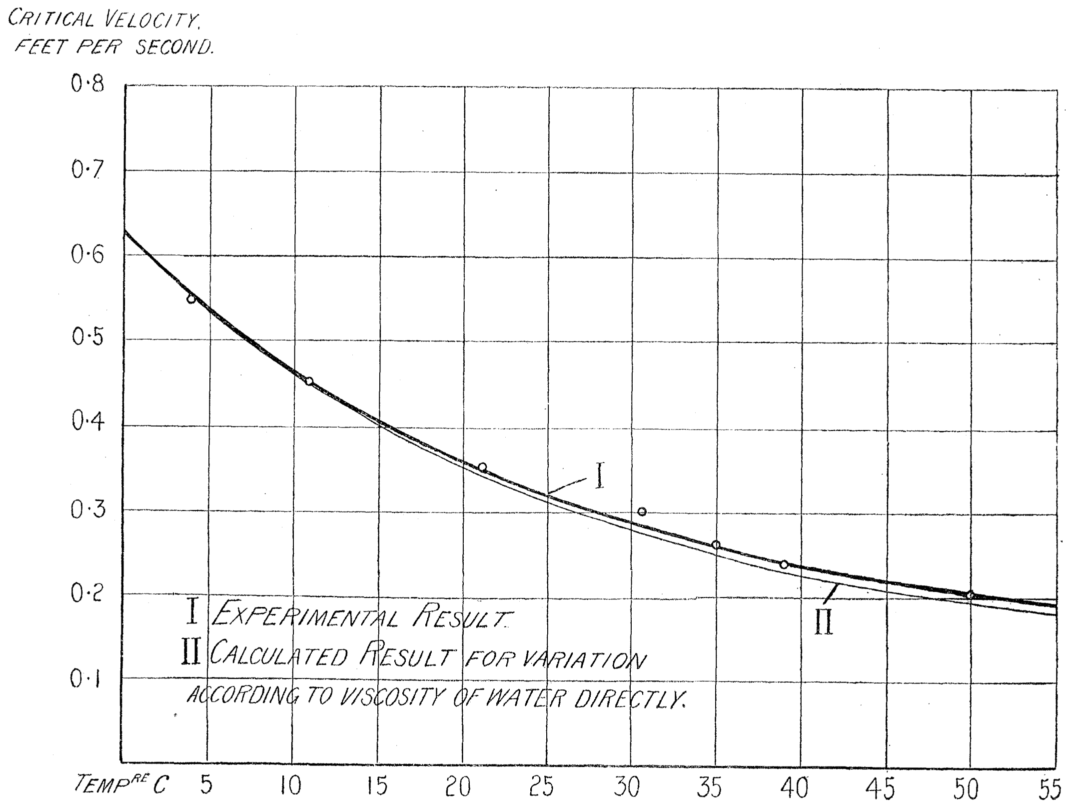


Fig. 6.

stream-line flow (shown dotted); the points of meeting are found to lie very approximately upon the straight line AB.

It is therefore apparent that these intersections vary as the viscosity, and they afford a verification of the formula.

This is brought out clearly by fig. 6, in which the velocities so found are plotted directly with respect to temperature. As will be seen, less weight is given to the observations in the neighbourhood of 30°, for the experimental results on stream-line motion are probably less accurate, as was pointed out in a previous section. The law of variation is found to be approximately

$$v_c^{-1} \propto 1 + 0{\cdot}03368\mathrm{T} + {\cdot}000156\mathrm{T}^2,$$

which agrees very closely with the variation in the viscosity of water, viz.,

$$\mu^{-1} \propto 1 + 0{\cdot}03368\mathrm{T} + {\cdot}000221\mathrm{T}^2.$$

It may therefore be concluded that for small pipes, over the range of temperature examined, the critical velocity of water varies directly as the viscosity.

In conclusion the authors desire to thank Professor BOVEY for placing the hydraulic laboratory of McGill University at their disposal, and also Dr. BARNES, who gave much assistance during the progress of the work.

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